10. Odmocnina z komplexního čísla, binomické rovnice

Př.) Řešte v C: $z^n = a$ $\quad n \in N;\ a, z \in C$; z ... neznámá

1. $a = 0 \Rightarrow z^n = 0 \Rightarrow z = 0$, tedy $z = \sqrt[n]{0} = 0$
2. $a \neq 0 \Rightarrow$ $\underline{z}$, $\underline{a}$ lze zapsat v goniom. tvaru $z = |z|(\cos\varphi + i\sin\varphi)$ $\quad a = |a|(\cos\alpha + i\sin\alpha)$

 - dos. do rce $z^n = a$

$$\left[|z|(\cos\varphi + i\sin\varphi)\right]^n = |a|(\cos\alpha + i\sin\alpha)$$

$$|z|^n(\cos n\varphi + i\sin n\varphi) = |a|(\cos\alpha + i\sin\alpha)$$

 - z rovnosti kompl. čísel: abs. hodnoty se rovnají, úhly (argumenty) se liší o celočíselné násobky 2π ($360°$)

$$|z|^n = |a| \quad \Rightarrow \quad |z| = \sqrt[n]{|a|}$$ [n-tá odmoc. abs. h. $\underline{a}$ – reál. číslo)

$$n\varphi = \alpha + 2k\pi \quad \Rightarrow \quad \varphi = \frac{\alpha + 2k\pi}{n} \qquad \left(\varphi = \frac{\alpha + k.360°}{n}\right)$$

$$(n\varphi = \alpha + k\cdot 360°)$$

kde $k = 0, 1, 2, \dots, n-1$

vzhledem k periodicitě fcí sin, cos stačí za $\underline{k}$ dos. jen čísla od 0 do $(n-1)$, pro $k = n$ dostaneme stejný výsledek jako pro $k = 0$

 - tedy $$z_k = \sqrt[n]{|a|}\left(\cos\frac{\alpha + 2k\pi}{n} + i\sin\frac{\alpha + 2k\pi}{n}\right)$$

$k = 0, 1, \dots, n-1$

- KOMPLEXNÍ n-tá ODMOCNINA ($n \in N$) z kompl. čísla $\underline{a}$ je každé číslo $\underline{z}$, pro které platí: $z^n = a$

- je-li $a = |a|(\cos\alpha + i\sin\alpha) \neq 0$, $n \in N$, pak EXISTUJE PRÁVĚ $\underline{n}$ KOMPLEXNÍCH ČÍSEL $\underline{z_k}$, která jsou komplexní n-tou odmocninou čísla $\underline{a}$ (tj. $z^n = a$)

$$z_k = \sqrt[n]{|a|}\left(\cos\frac{\alpha + 2k\pi}{n} + i\sin\frac{\alpha + 2k\pi}{n}\right)$$ α ... v rad $\quad k = 0, 1, \dots, n-1$

$$\left[z_k = \sqrt[n]{|a|}\left(\cos\frac{\alpha + k.360°}{n} + i\sin\frac{\alpha + k.360°}{n}\right)\right.$$ α ... ve stupních]

- PRO PRAKTICKÉ VÝPOČTY vhodné argumenty upravit: $\varphi = \frac{\alpha + 2k\pi}{n} = \frac{\alpha}{n} + \frac{k\cdot 2\pi}{n}$

$$\left[\varphi = \frac{180° + k.360°}{3} = 60° + k\,120°\right]$$

Příklady

① Urči $\sqrt[3]{-8}$ $\quad z_k = \sqrt[3]{-8}$ $\quad a = -8 = 8(\cos\pi + i\sin\pi) = 8(\cos 180° + i\sin 180°)$ (na goniom. tvar)

$n = 3 \Rightarrow k = 0, 1, 2$ $\quad [n - 1 = 3 - 1 = 2]$

[$\alpha = 180°$ $\quad |-8| = 8$]

VZOREC $$z_k = \sqrt[n]{|a|}\left(\cos\frac{\alpha + k.360°}{n} + i\sin\frac{\alpha + k.360°}{n}\right) \qquad k = 0, 1, 2, \dots, n-1$$

(*) $$z_k = \sqrt[3]{8}\left(\cos\frac{180° + k.360°}{3} + i\sin\frac{180° + k.360°}{3}\right)$$

(**) $$z_k = 2\left(\cos(60° + k\cdot 120°) + i\sin(60° + k.120°)\right) \qquad k = 0, 1, 2$$

dos. do (**) – VÝHODNĚJŠÍ

$k = 0$: $\quad z_0 = 2\left[\cos(60° + 0.120°) + i\sin(60° + 0.120°)\right] = 2(\underbrace{\cos 60°}_{\frac{1}{2}} + i\underbrace{\sin 60°}_{\frac{\sqrt{3}}{2}}) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = \underline{1 + \sqrt{3}i}$

$k = 1$: $\quad z_1 = 2\left[\cos(60° + 1.120°) + i\sin(60° + 1.120°)\right] = 2(\underbrace{\cos 180°}_{-1} + i\underbrace{\sin 180°}_{0}) = 2(-1 + 0i) = \underline{-2}$

$k = 2$: $\quad z_2 = 2\left[\cos(60° + 2\cdot 120°) + i\sin(60° + 2.120°)\right] = 2(\cos 300° + i\sin 300°) =$

$= 2(\underbrace{\cos 60°}_{\frac{1}{2}} - i\underbrace{\sin 60°}_{\frac{\sqrt{3}}{2}}) = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \underline{1 - \sqrt{3}i}$ $\quad$ [+ IV. 60° [α = 360° − 300°]]

[dos. do (*) – dostáváme větší úhly $\quad z_k = \sqrt[3]{8}\left(\cos\frac{180°+k\cdot 360°}{3} + i\sin\frac{180°+k\cdot 360°}{3}\right)$

$k=0:\quad z_0 = 2\left(\cos\frac{180°}{3} + i\sin\frac{180°}{3}\right) = 2(\cos 60° + i\sin 60°) = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = \underline{1+\sqrt{3}i}$

$k=1:\quad z_1 = 2\left(\cos\frac{180°+1\cdot 360°}{3} + i\sin\frac{180°+1\cdot 360°}{3}\right) = 2\left(\cos\frac{540°}{3} + i\sin\frac{540°}{3}\right) =$

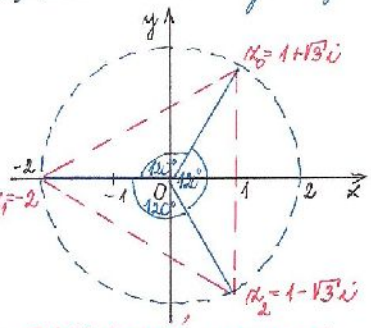
$= 2(\underbrace{\cos 180°}_{-1} + i\underbrace{\sin 180°}_{0}) = 2(-1+0i) = \underline{-2}$

$k=2:\quad z_2 = 2\left(\cos\frac{180°+2\cdot 360°}{3} + i\sin\frac{180°+2\cdot 360°}{3}\right) = 2\left(\cos\frac{900°}{3} + i\sin\frac{900°}{3}\right)$

$= 2(\cos 300° + i\sin 300°) = 2(\cos 60° - i\sin 60°) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \underline{1-\sqrt{3}i}$]

[+ IV. kv. 60° −]

b) nakreslete obrazy n-tých komplex. odmocnin v Gaussově rovině



- obrazy n-tých komplexních odmocnin z čísla a (pro $n \geq 3$) jsou VRCHOLY PRAVIDELNÉHO n-ÚHELNÍKU VEPSANÉHO DO KRUŽNICE se středem v počátku $O[0,0]$ a $r = \sqrt[n]{|a|}$
- známe-li jednu odmocninu => ostatní lze určit graficky
- v našem případě: ROVNOSTRANNÝ Δ vepsaný do $k(O;\ r=2)$

- BINOMICKÉ ROVNICE
  - rovnice tvaru $px^n + q = 0 \quad p \neq 0 \quad p,q \in \mathbb{C} \quad n \in \mathbb{N} \quad x \in \mathbb{C}$ neznámá
  - řešením jsou n-té komplexní odmocniny čísla $-\frac{q}{p}$

  $\left[\ px^n + q = 0 \Rightarrow px^n = -q \Rightarrow x^n = -\frac{q}{p} \Rightarrow x = \sqrt[n]{-\frac{q}{p}}\ \right]$

Příklady

① Řešte v C: $x^6 - 1 = 0 \qquad O = \mathbb{C} \quad D = \mathbb{C}$

$x^6 = 1$

$x = \sqrt[6]{1} \qquad 1 = \cos 0 + i\sin 0$ $\qquad$ [y, 1, x $\quad \varphi = 0 \quad |a| = |1| = 1 \quad \underline{1 = \cos 0 + i\sin 0}$]

a na goniom. tvar

$x_k = \sqrt[n]{|a|}\left(\cos\frac{\alpha+2k\pi}{n} + i\sin\frac{\alpha+2k\pi}{n}\right)$

(6-1)

$x_k = \sqrt[6]{1}\left(\cos\frac{0+2k\pi}{6} + i\sin\frac{0+2k\pi}{6}\right) = 1\left(\cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3}\right) \qquad k = 0,1,2,3,4,5$

- postupně dosaz. za $k = 0,1,2,3,4,5$ $\qquad = (\cos k\cdot 60° + i\sin k\cdot 60°)$

$k=0:\quad x_0 = \underbrace{\cos 0}_{1} + i\underbrace{\sin 0}_{0} = \underline{1}$

$k=1:\quad x_1 = \cos 60° + i\sin 60° = \underline{\frac{1}{2} + \frac{\sqrt{3}}{2}i}$

$k=2:\quad x_2 = \cos 120° + i\sin 120° = -\cos 60° + i\sin 60° = \underline{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}$

[− II. kv. α' = 60° +]

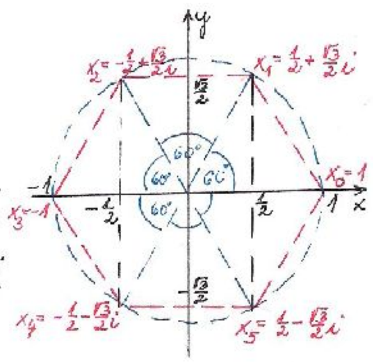
$k=3:\quad x_3 = \cos 3\cdot 60° + i\sin 3\cdot 60° = \underbrace{\cos 180°}_{-1} + i\underbrace{\sin 180°}_{0} = \underline{-1}$

$k=4:\quad x_4 = \cos 240° + i\sin 240° = -\cos 60° - i\sin 60° = \underline{-\frac{1}{2} - \frac{\sqrt{3}}{2}i}$

[− III. kv. α' = 60° −]

$k=5:\quad x_5 = \cos 300° + i\sin 300° = \cos 60° - i\sin 60° = \underline{\frac{1}{2} - \frac{\sqrt{3}}{2}i}$

[+ IV. kv. α' = 60° −]



POZNÁMKA

- některé binom. rce lze řešit i převedením na SOUČINOVÝ TVAR   více viz další článek

$x^6-1=0$

$(x^3)^2-1^2=0$

a^2-b^2

$(x^3-1)(x^3+1)=0$

$(a-b)(a+b)$

$(x-1)(x^2+x+1)(x+1)(x^2-x+1)=0$

$x_1=1$ $x^2+x+1=0$ $x_4=-1$ $x^2-x+1=0$

$x_{2,3}=\frac{-1\pm\sqrt{1-4}}{2}$ $x_{5,6}=\frac{1\pm\sqrt{1-4}}{2}$

$x_{2,3}=\frac{-1\pm\sqrt{-3}}{2}$ $x_{5,6}=\frac{1\pm\sqrt{-3}}{2}$

$x_{2,3}=\frac{-1\pm\sqrt{3i^2}}{2}$ $x_{5,6}=\frac{1\pm\sqrt{3i^2}}{2}$

$x_{2,3}=\frac{-1\pm i\sqrt3}{2}$ $x_{5,6}=\frac{1\pm i\sqrt3}{2}$

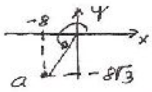
③ Řešte v C

a) $z^4+8+8\sqrt3 i=0$ $\Gamma=\mathbb{C}$ $D=C$

$z^4=-8-8\sqrt3 i$

$z=\sqrt[4]{-8-8\sqrt3 i}$

$a=-8-8\sqrt3 i=[-8,-8\sqrt3]$



$\alpha \dots$ III. kv.

na gon. tvar

$|a|=\sqrt{(-8)^2+(-8\sqrt3)^2}=\sqrt{64+64\cdot3}=\sqrt{64+192}=\sqrt{256}=16$

$\sin\alpha=\frac{-8\sqrt3}{16}=-\frac{\sqrt3}{2}\Rightarrow\alpha'=60^\circ\Rightarrow$ III. kv. $\alpha=180^\circ+\alpha'$

$\alpha=180^\circ+60^\circ=240^\circ$

$a=16(\cos240^\circ+i\sin240^\circ)$

$z_k=\sqrt[n]{|a|}\left(\cos\frac{\alpha+k.360^\circ}{n}+i\sin\frac{\alpha+k.360^\circ}{n}\right)$

$z_k=\sqrt[4]{16}\left(\cos\frac{240^\circ+k.360^\circ}{4}+i\sin\frac{240^\circ+k.360^\circ}{4}\right)$

$z_k=2\left(\cos(60^\circ+k\cdot90^\circ)+i\sin(60^\circ+k\cdot90^\circ)\right)$ $k=0,1,2,3$

$k=0$: $z_0=2(\cos60^\circ+i\sin60^\circ)=2\left(\frac12+\frac{\sqrt3}{2}i\right)=1+\sqrt3 i$

$k=1$: $z_1=2\left[\cos(60^\circ+90^\circ)+i\sin(60^\circ+90^\circ)\right]=2(\cos150^\circ+i\sin150^\circ)=$ [II. kv. ⊖ $\alpha'=30^\circ$ ⊕]

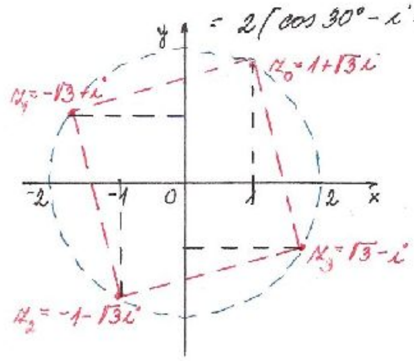
$=2(-\cos30^\circ+i\sin30^\circ)=2\left(-\frac{\sqrt3}{2}+\frac12 i\right)=-\sqrt3+i$

$k=2$: $z_2=2\left[\cos(60^\circ+2\cdot90^\circ)+i\sin(60^\circ+2\cdot90^\circ)\right]=2(\cos240^\circ+i\sin240^\circ)=$ [III. kv. ⊖ $\alpha'=60^\circ$ ⊖]

$=2(-\cos60^\circ-i\sin60^\circ)=2\left(-\frac12-\frac{\sqrt3}{2}i\right)=-1-\sqrt3 i$

$k=3$: $z_3=2\left[\cos(60^\circ+3\cdot90^\circ)+i\sin(60^\circ+3\cdot90^\circ)\right]=2(\cos330^\circ+i\sin330^\circ)=$ [IV. kv. ⊕ $\alpha'=30^\circ$ ⊖]

$=2(\cos30^\circ-i\sin30^\circ)=2\left(\frac{\sqrt3}{2}-\frac12 i\right)=\sqrt3-i$



b) $27x^3+125=0 \qquad \mathcal{O}=\mathbb{C} \qquad \mathcal{D}=\mathbb{C}$

$$27x^3=-125$$
$$x^3=-\frac{125}{27}$$
$$x=\sqrt[3]{-\frac{125}{27}}$$

$\left[\ a=-\frac{125}{27}\right.$ na gon. trav $-\frac{125}{27}$ $\quad |a|=\left|-\frac{125}{27}\right|=\frac{125}{27}$, $\quad \alpha=180^\circ$

$\left. a=\frac{125}{27}(\cos 180^\circ+i\sin 180^\circ)\ \right]$

$$x_k=\sqrt[3]{\frac{125}{27}}\left(\cos\frac{180^\circ+k\cdot 360^\circ}{3}+i\sin\frac{180^\circ+k\cdot 360^\circ}{3}\right) \qquad k=0,1,2$$

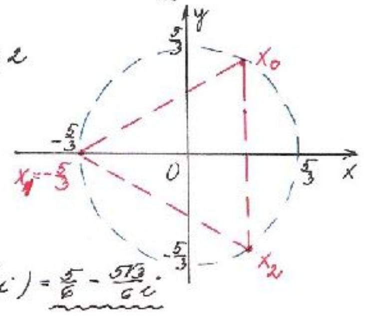
$$x_k=\frac{5}{3}\left[\cos(60^\circ+k\cdot 120^\circ)+i\sin(60^\circ+k\cdot120^\circ)\right]$$

$$x_0=\frac{5}{3}(\cos 60^\circ+i\sin 60^\circ)=\frac{5}{3}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=\frac{5}{6}+\frac{5\sqrt{3}}{6}i$$

$$x_1=\frac{5}{3}(\underbrace{\cos 180^\circ}_{-1}+i\underbrace{\sin 180^\circ}_{0})=\frac{5}{3}(-1+0i)=-\frac{5}{3}$$

$$x_2=\frac{5}{3}(\cos 300^\circ+i\sin 300^\circ)=\frac{5}{3}(\cos 60^\circ+i\sin 60^\circ)=\frac{5}{3}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=\frac{5}{6}-\frac{5\sqrt{3}}{6}i$$

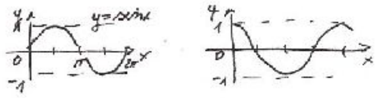
[IV. kv. ⊕ $\alpha'=60^\circ$ ⊖]



c) $16x^4-1=0 \qquad \mathcal{O}=\mathbb{C},\ \mathcal{D}=\mathbb{C}$

$$16x^4=1$$
$$x^4=\frac{1}{16}$$
$$x=\sqrt[4]{\frac{1}{16}}$$

$\left[\ a=\frac{1}{16}=\frac{1}{16}(\cos 0^\circ+i\sin 0^\circ)\right.$ na gon. trav $\frac{1}{16}$ $\quad \left|\frac{1}{16}\right|=\frac{1}{16}$, $\left.\alpha=0^\circ\ \right]$



$$x_k=\sqrt[4]{\frac{1}{16}}\left(\cos\frac{0+k\cdot 360^\circ}{4}+i\sin\frac{0+k\cdot 360^\circ}{4}\right)$$

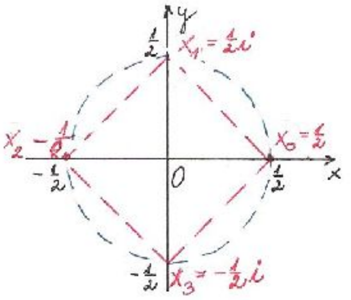
$$x_k=\frac{1}{2}(\cos k\cdot 90^\circ+i\sin k\cdot 90^\circ) \qquad k=0,1,2,3$$

$$x_0=\frac{1}{2}(\cos 0^\circ+i\sin 0^\circ)=\frac{1}{2}(1+0i)=\frac{1}{2}$$

$$x_1=\frac{1}{2}(\cos 90^\circ+i\sin 90^\circ)=\frac{1}{2}(0+i)=\frac{1}{2}i$$

$$x_2=\frac{1}{2}(\cos 180^\circ+i\sin 180^\circ)=\frac{1}{2}(-1+0i)=-\frac{1}{2}$$

$$x_3=\frac{1}{2}(\cos 270^\circ+i\sin 270^\circ)=\frac{1}{2}(0-i)=-\frac{1}{2}i$$



d) $x^3+i=0$

$$x^3=-i$$
$$x=\sqrt[3]{-i}$$

$\left[\ -i=1(\cos 270^\circ+i\sin 270^\circ)\right.$ $\quad |-i|=1$, $\left.\alpha=270^\circ\ \right]$

$$x_k=\sqrt[3]{1}\left(\cos\frac{270^\circ+k\cdot 360^\circ}{3}+i\sin\frac{270^\circ+k\cdot 360^\circ}{3}\right)$$

$$x_k=1\left[\cos(90^\circ+k\cdot 120^\circ)+i\sin(90^\circ+k\cdot 120^\circ)\right] \qquad k=0,1,2$$

$$x_0=\underbrace{\cos 90^\circ}_{0}+i\underbrace{\sin 90^\circ}_{+1}=+i$$

$$x_1=\cos(90^\circ+120^\circ)+i\sin(90^\circ+120^\circ)=\cos 210^\circ+i\sin 210^\circ=$$

[III. kv. ⊖ 30° ⊖]

$$=-\cos 30^\circ-i\sin 30^\circ=-\frac{\sqrt{3}}{2}-\frac{1}{2}i$$

$$x_2=\cos(90^\circ+2\cdot 120^\circ)+i\sin(90^\circ+2\cdot120^\circ)=$$

$$=\cos 330^\circ+i\sin 330^\circ=\cos 30^\circ-i\sin 30^\circ=$$

[IV. kv. ⊕ 30° ⊖]

$$=\frac{\sqrt{3}}{2}-\frac{1}{2}i$$